Learning Probably Approximately Correct Maximin Strategies in Games with Infinite Strategy Spaces [*]

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Abstract

We tackle the problem of learning equilibria in *simulation-based* games. In such games, the players' utility functions cannot be described analytically, as they are given through a black-box simulator that can be queried to obtain noisy estimates of the utilities. This is the case in many real-world games in which a complete description of the elements involved is not available upfront, such as complex military settings and online auctions. In these situations, one usually needs to run costly simulation processes to get an accurate estimate of the game outcome. As a result, solving these games begets the challenge of designing learning algorithms that can find (approximate) equilibria with high confidence, using as few simulator queries as possible. We focus on two-player zero-sum games with *infinite strategy spaces*. Drawing from the best arm identification literature, we design two algorithms with theoretical guarantees to learn *maximin* strategies in these games. The first one works in the *fixed-confidence* setting, guaranteeing the desired confidence level while minimizing the number of queries. Instead, the second algorithm fits the *fixed-budget* setting, maximizing the confidence without exceeding the given maximum number of queries. First, we formally prove δ-PAC theoretical guarantees for our algorithms under some regularity assumptions, which are encoded by letting the utility functions be drawn from a Gaussian process. Then, we experimentally evaluate our techniques on a testbed made of randomly generated games and instances representing simple real-world security settings.

Introduction

Over the last two decades, game-theoretic models have received a growing interest from the AI community, as they allow to design artificial agents endowed with the ability of reasoning strategically in complex multi-agent settings. This surge of interest was driven by many successful applications of game theory to challenging real-world problems, such as building robust protection strategies in security domains (Tambe 2011), designing truthful auctions for web advertising (Gatti et al. 2015), and solving large zero-sum recreational games, *e.g.*, Go (Silver et al. 2016), different variants of Poker (Brown and Sandholm 2018, 2019), and Bridge (Rong, Qin, and An 2019).

Most of the game-theoretic studies in AI focus on models where a complete description of the game is available, *i.e.*, the players' utilities can be expressed analytically. This is the case of recreational games, which are commonly used as benchmarks for evaluating algorithms to compute equilibria in games (Brown and Sandholm 2017). However, in many real-world problems, the players' utilities may *not* be readily available, as they are the outcome of a complex process governed by unknown parameters. This is the case, *e.g.*, in complex military settings where a comprehensive description of the environment and the units involved is not available, and online auctions in which the platform owner does not have complete knowledge of the parties involved. These scenarios can be addressed with *simulation-based games* (SBGs) (Vorobeychik and Wellman 2009), where the players' utilities are expressed by means of a black-box simulator that, given some players' strategies, can be queried to obtain a noisy estimate of the utilities obtained when playing such strategies. These models beget new challenges in the design of algorithms to solve games: *(i)* they have to learn (approximate) equilibria by using only noisy observations of the utilities, and *(ii)* they should use as few queries as possible, since running the simulator is usually a costly operation. Additionally, using the simulator while playing the game is unfeasible, since the simulation process might be prohibitively time consuming, as it is the case, *e.g.*, in military settings where the units have to take prompt decisions when on the battlefield. Thus, the algorithms must first perform a pure exploration learning phase and, then, use the (approximate) equilibrium learned to play the game.

Despite the modeling power of SBGs, recent works studying such games are only sporadic, addressing specific settings such as, *e.g.*, symmetric games with a large number of players (Wiedenbeck, Yang, and Wellman 2018; Sokota, Ho, and Wiedenbeck 2019), empirical mechanism design (Viqueira et al. 2019), and two-player zero-sum finite games (Garivier, Kaufmann, and Koolen 2016). To the best of our knowledge, the majority of these works focus on the case in which each player has a finite number of strategies

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available. However, in most of the game settings in which simulations are involved, the players have an infinite number of choices available, *e.g.*, physical quantities, such as angle of movement and velocity of units on a military field, bids in auctions, and trajectories in robot planning. Dealing with infinite strategies leads to further challenges in the design of learning algorithms, since, being a complete exploration of the strategy space unfeasible, providing strong theoretical guarantees is, in general, a non-trivial task.

Original Contributions We study the problem of learning equilibria in *two-player zero-sum* SBGs with *infinite strategy spaces*, providing theoretical guarantees. Specifically, we focus on *maximin* strategies for the first player, *i.e.*, those maximizing her utility under the assumption that the second player acts so as to minimize it, after observing the first player's course of play. For instance, this is the case in security games where a terrestrial counter-air defensive unit has to shoot an heat-seeking missile to a moving target that represents an approaching enemy airplane, which, after the attack has started, can respond to it by deploying an obfuscating flare with the intent of deflecting the missile trajectory. When dealing with infinite strategy spaces, some regularity assumptions on the players' utilities are in order, since, otherwise, one cannot design learning algorithms with provable theoretical guarantees. In this work, we encode our regularity assumptions on the utility function by modeling it as a sample from a *Gaussian process* (GP) (Williams and Rasmussen 2006). We design two algorithms able to learn (approximate) maximin strategies in two-player zero-sum SBGs with infinite strategy spaces, drawing from techniques used in the best arm identification literature. The first algorithm we propose, called M-GP-LUCB, is for the *fixed-confidence* setting, where the objective is to find an (approximate) maximin strategy with a given (high) confidence, using as few simulator queries as possible. Instead, the second algorithm, called SE-GP, is for the *fixed-budget* setting, in which a maximum number of queries is given in advance, and the task is to return an (approximate) maximin strategy with confidence as high as possible. First, we prove δ-PAC (*i.e.*, *probably approximately correct*) theoretical guarantees for our algorithms in the easiest setting in which the strategy spaces are finite. Then, we show how these results can be generalized to SBGs with infinite strategy spaces by leveraging the GP assumption. Finally, we experimentally eventuate our algorithms on a testbed made of randomly generated games and instances based on the missile-airplane security game described above. For SBGs with finite strategy spaces, we also compare our algorithms with the M-LUCB algorithm introduced by (Garivier, Kaufmann, and Koolen 2016) (the current state-of-the-art method for learning maximin strategies in two-player zero-sum finite games), showing that our methods dramatically outperform it.

Preliminaries

A *two-player zero-sum game with infinite strategy spaces* is a tuple $\Gamma = (\mathcal{X}, \mathcal{Y}, u)$, where $\mathcal{X} \subset \mathbb{R}^d$ and $\mathcal{Y} \subset \mathbb{R}^d$ are compact and convex sets of strategies available to the first and the second player, respectively, while $u : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ is a function defining the utility for the first player. [1] Since the game is zero-sum, the second player's utility is given by $-u$. A *two-player zero-sum game with finite strategy spaces* is defined analogously, with $\mathcal{X}$ and $\mathcal{Y}$ being finite sets, *i.e.*, $\mathcal{X} := \{x^1, \dots, x^n\}$ and $\mathcal{Y} := \{y^1, \dots, y^m\}$, with $n > 1$ and $m > 1$ denoting the finite numbers of strategies available to the first and the second player, respectively. For the ease of notation, letting $\Pi := \mathcal{X} \times \mathcal{Y}$, we denote with $\boldsymbol{\pi} := (x, y) \in \Pi$ a *strategy profile*, *i.e.*, a tuple specifying a strategy $x \in \mathcal{X}$ for the first player and a strategy $y \in \mathcal{Y}$ for the second player.

In this work, we are concerned with the computation of *maximin* strategies, adopting the perspective of the first player. In words, we seek for a first player's strategy that maximizes her utility, assuming a worst-case opponent that acts so as to minimize it. Since the game is zero sum, we can assume that the second player decides how to play after observing the first player's move, and, thus, playing a maximin strategy is the best choice for the first player. [2] Formally, given a first player's strategy $x \in \mathcal{X}$, we denote with $y^*(x) \in \arg\min_{y \in \mathcal{Y}} u(x, y)$ a second player's best response to x. Then, $x^* \in \mathcal{X}$ is a maximin strategy for the first player if $x^* \in \arg\max_{x \in \mathcal{X}} u(x, y^*(x))$, with $\boldsymbol{\pi}^* := (x^*, y^*(x^*))$ denoting its corresponding maximin strategy profile.

Simulation-Based Games In SBGs, the utility function u is not readily available, but it is rather specified by an exogenous simulator that provides noisy point estimates of it. As a result, in SBGs, one cannot explicitly compute a maximin strategy, and, thus, the problem is to learn one by sequentially querying the simulator. At each round t, the simulator is given a strategy profile $\boldsymbol{\pi}_t \in \Pi$ and returns an estimated utility $\tilde{u}_t := u(\boldsymbol{\pi}_t) + e_t$, where $e_t \sim \mathcal{N}(0, \lambda)$ is i.i.d. Gaussian noise. The goal is to find a good approximation (see Equation (1)) of a maximin strategy $x^* \in \mathcal{X}$ as rapidly as possible, *i.e.*, limiting the number of queries to the simulator. To achieve this, we follow the approach of Garivier, Kaufmann, and Koolen (2016) and propose some *dynamic querying algorithms* (see Algorithm 1, where $\textsc{Sim}(\boldsymbol{\pi})$ is a simulator query for $\boldsymbol{\pi} \in \Pi$), which are characterized by:

- a querying rule that indicates which strategy profile $\boldsymbol{\pi}_t \in \Pi$ is sent as input to the simulator at each round t;
- a stopping rule that determines the round T after which the algorithm terminates its execution;
- a final guess $\overline{\boldsymbol{\pi}} := (\bar{x}, \bar{y}) \in \Pi$ for the (true) maximin strategy profile $\boldsymbol{\pi}^*$ of the game.

Given a desired approximation $\epsilon \geq 0$, the objective of the algorithm is to find an ϵ-maximin strategy with high accuracy, using as few queries as possible to the simulator. Formally, given $\delta \in (0, 1)$, our goal is to design algorithms that are δ-PAC, *i.e.*, they satisfy:

$$\forall u \quad \mathbb{P}\Big(|u(\boldsymbol{\pi}^*) - u(\bar{x}, y^*(\bar{x}))| \leq \epsilon \Big) \geq 1 - \delta, \qquad (1)$$

[1]For the ease of presentation, in the following we focus on the case in which $\mathcal{X} \subset [0, 1]$ and $\mathcal{Y} \subset [0, 1]$ are closed intervals.

[2]This assumption is in line with the classical Stackelberg model in which the second player (follower) gets to play after observing the strategy of the first one (leader) (von Stackelberg 1934).

Algorithm 1 Dynamic Querying Algorithm

```
1: $t \leftarrow 1$
2: **do**
3:     Select a strategy profile $\boldsymbol{\pi}_t \in \Pi$ according
           to the querying rule
4:     Get estimated utility $\tilde{u}_t \leftarrow \textsc{Sim}(\boldsymbol{\pi}_t)$
5:     Update the algorithm parameters using $\tilde{u}_t$
6:     $t \leftarrow t + 1$
7: **while** stopping condition is not met
8: **return** final guess $\overline{\boldsymbol{\pi}} = (\bar{x}, \bar{y})$ for the maximin profile
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while keeping the number of rounds T as small as possible. This is known as the *fixed-confidence* setting. An alternative is to consider the *fixed-budget* case, where the maximum number of rounds T is given in advance, and the goal is to minimize the probability δ that $\bar{x}$ is not an ϵ-maximin strategy. Notice that, for SBGs with finite strategy spaces, the δ-PAC property in Equation (1) can only require $u(\boldsymbol{\pi}^*) - u(\bar{x}, y^*(\bar{x})) \leq \epsilon$, since $u(\boldsymbol{\pi}^*) > u(\bar{x}, y^*(\bar{x}))$.

Gaussian Processes To design δ-PAC algorithms working with SBGs having infinite strategy spaces, we first need to introduce some regularity assumptions on the utility functions u. In this work, we model the utility as a sample from a GP, which is a collection of dependent random variables, one for each action profile $\boldsymbol{\pi} \in \Pi$, every finite subset of which is multivariate Gaussian distributed (Williams and Rasmussen 2006). A GP$(\mu(\boldsymbol{\pi}), k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ is fully specified by its *mean* function $\mu : \Pi \mapsto \mathbb{R}$, with $\mu(\boldsymbol{\pi}) := \mathbb{E}[u(\boldsymbol{\pi})]$, and its *covariance* (or *kernel*) function $k : \Pi \times \Pi \mapsto \mathbb{R}$, with $k(\boldsymbol{\pi}, \boldsymbol{\pi}') := \mathbb{E}[(u(\boldsymbol{\pi}) - \mu(\boldsymbol{\pi}))(u(\boldsymbol{\pi}') - \mu(\boldsymbol{\pi}'))]$. W.l.o.g., we assume that $\mu \equiv \mathbf{0}$ and the variance is bounded, *i.e.*, $k(\boldsymbol{\pi}, \boldsymbol{\pi}) := \sigma^2 \leq 1$ for every $\boldsymbol{\pi} \in \Pi$. Note that the GP assumption guarantees that the utility function u has a certain degree of smoothness, without relying on rigid parametric assumptions, such as linearity. Intuitively, the kernel function k determines the correlation of the utility values across the space of strategy profiles Π, thus encoding the smoothness properties of the utility functions u sampled from GP$(\mu(\boldsymbol{\pi}), k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ (for some examples of commonly used kernels, see Section).

We also need GPs in our algorithms, as they use GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ as prior distribution over u. The major advantage of working with GPs is that they admit simple analytical formulas for the mean and covariance of the posterior distribution. These relations can be easily expressed using matrix notation, as follows. Let $\tilde{\mathbf{u}}_t := [\tilde{u}_1, \ldots, \tilde{u}_t]^\top$ be the vector of utility values observed up to round t, obtained by querying the simulator on the strategy profiles $\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_t$. Then, the posterior distribution over u is still a GP, with mean $\mu_t(\boldsymbol{\pi})$, covariance $k_t(\boldsymbol{\pi}, \boldsymbol{\pi}')$, and variance $\sigma_t^2(\boldsymbol{\pi})$, which are defined as follows:

$$\mu_t(\boldsymbol{\pi}) := \mathbf{k}_t(\boldsymbol{\pi})^\top \left(K_t + \lambda I\right)^{-1} \tilde{\mathbf{u}}_t, \tag{2}$$

$$k_t(\boldsymbol{\pi}, \boldsymbol{\pi}') := k(\boldsymbol{\pi}, \boldsymbol{\pi}') - \mathbf{k}_t(\boldsymbol{\pi})^\top (K_t + \lambda I)^{-1} \mathbf{k}_t(\boldsymbol{\pi}'), \tag{3}$$

$$\sigma_t^2(\boldsymbol{\pi}) := k_t(\boldsymbol{\pi}, \boldsymbol{\pi}), \tag{4}$$

where $\mathbf{k}_t(\boldsymbol{\pi}) := [k(\boldsymbol{\pi}, \boldsymbol{\pi}_1), \ldots, k(\boldsymbol{\pi}, \boldsymbol{\pi}_t)]^\top$ and K_t is the positive definite $t \times t$ kernel matrix, whose (i, j)-th entry is $k(\boldsymbol{\pi}_i, \boldsymbol{\pi}_j)$. The posterior parameters update formulas can also be expressed recursively, thus avoiding costly matrix inversions, as shown in (Chowdhury and Gopalan 2017). Letting $\boldsymbol{\pi}_t$ and $\tilde{u}_t$ be, respectively, the queried strategy profile and the observed utility at round t, we can write:

$$\mu_t(\boldsymbol{\pi}) \leftarrow \mu_{t-1}(\boldsymbol{\pi}) + \frac{k_{t-1}(\boldsymbol{\pi}, \boldsymbol{\pi}_t)}{\lambda + \sigma_{t-1}^2(\boldsymbol{\pi}_t)} (\tilde{u}_t - \mu_{t-1}(\boldsymbol{\pi}_t)), \tag{5}$$

$$k_t(\boldsymbol{\pi}, \boldsymbol{\pi}') \leftarrow k_t(\boldsymbol{\pi}, \boldsymbol{\pi}') - \frac{k_{t-1}(\boldsymbol{\pi}, \boldsymbol{\pi}_t) k_{t-1}(\boldsymbol{\pi}_t, \boldsymbol{\pi}')}{\lambda + \sigma_{t-1}^2(\boldsymbol{\pi}_t)}, \tag{6}$$

$$\sigma_t^2(\boldsymbol{\pi}) \leftarrow \sigma_{t-1}^2(\boldsymbol{\pi}) - \frac{k_{t-1}^2(\boldsymbol{\pi}, \boldsymbol{\pi}_t)}{\lambda + \sigma_{t-1}^2(\boldsymbol{\pi}_t)}. \tag{7}$$

At the beginning of the algorithms, estimates are initialized using the GP prior GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$, *i.e.*, formally, $\mu_0(\boldsymbol{\pi}) := \mathbf{0}$, $k_0(\boldsymbol{\pi}, \boldsymbol{\pi}') := k(\boldsymbol{\pi}, \boldsymbol{\pi}')$, and $\sigma_0^2(\boldsymbol{\pi}) := k(\boldsymbol{\pi}, \boldsymbol{\pi}) = \sigma^2$.

Fixed-Confidence Setting

In this section and the following one, we present our learning algorithms for the easiest setting of SBGs with finite strategy spaces. For the fixed-confidence setting, we propose a δ-PAC dynamic querying algorithm (called M-GP-LUCB, see Algorithm 2) based on the M-LUCB approach introduced by Garivier, Kaufmann, and Koolen (2016) and provide a bound on the number of rounds T_δ it requires, as a function of the confidence level δ. While our algorithm shares the same structure as M-LUCB, it uses confidence bounds relying on the GP assumption, and, thus, different proofs are needed to show its δ-PAC properties. Our algorithm and its theoretical guarantees have the crucial advantage of being easily generalizable to SBGs with infinite strategy spaces.

For every strategy profile $\boldsymbol{\pi} \in \Pi$, the algorithm keeps track of a confidence interval $[L_t(\boldsymbol{\pi}), U_t(\boldsymbol{\pi})]$ on $u(\boldsymbol{\pi})$ built using the utility values $\tilde{u}_t$ observed from the simulator up to round t. Using GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ as prior distribution over the utility function u, the lower bounds of the intervals are defined as $L_t(\boldsymbol{\pi}) := \mu_t(\boldsymbol{\pi}) - \sqrt{b_t}\sigma_t(\boldsymbol{\pi})$ and the upper bounds as $U_t(\boldsymbol{\pi}) := \mu_t(\boldsymbol{\pi}) + \sqrt{b_t}\sigma_t(\boldsymbol{\pi})$, where μ_t and σ_t^2 are the mean and the variance of the posterior distribution computed with observations up to round t (see Equations (2)–(4)), while b_t is an exploration term that depends from the context (see Theorem 1).

At the end of every even round t, the algorithm selects the strategy profiles to give as inputs to the simulator during the next two rounds $t + 1$ and $t + 2$. For every $x \in \mathcal{X}$, let

$$\gamma_t(x) := \operatorname*{argmin}_{y \in \mathcal{Y}} L_t(x, y)$$

be the second player's best response to x computed using the lower bounds L_t. Moreover, let

$$\bar{x}_t := \operatorname*{argmax}_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} \mu_t(x, y)$$

be the maximin strategy computed using the posterior mean μ_t. Then, in the following two rounds, the algorithm selects the strategy profiles $\boldsymbol{\pi}_{t+1}$ and $\boldsymbol{\pi}_{t+2}$, defined as follows:

$$\boldsymbol{\pi}_{t+1} := (\bar{x}_t, \gamma_t(\bar{x}_t)) \tag{8}$$

$$\boldsymbol{\pi}_{t+2} := \operatorname*{argmax}_{\boldsymbol{\pi} \in \{(x, \gamma_t(x))\}_{x \neq \bar{x}_t}} U_t(\boldsymbol{\pi}). \tag{9}$$

Algorithm 2 M-GP-LUCB(ϵ, δ)

```
1: Initialize $t \leftarrow 0$, $\mu_0(\boldsymbol{\pi}) \leftarrow \mathbf{0}$, $k_0(\boldsymbol{\pi}, \boldsymbol{\pi}') \leftarrow k(\boldsymbol{\pi}, \boldsymbol{\pi}')$
2: do
3:     Select $\boldsymbol{\pi}_{t+1}$ and $\boldsymbol{\pi}_{t+2}$ using Eqs. (8)–(9)
4:     $\tilde{u}_{t+1} \leftarrow \textsc{Sim}(\boldsymbol{\pi}_{t+1})$, $\tilde{u}_{t+2} \leftarrow \textsc{Sim}(\boldsymbol{\pi}_{t+2})$
5:     Compute $\mu_{t+2}(\boldsymbol{\pi})$ and $k_{t+2}(\boldsymbol{\pi}, \boldsymbol{\pi}')$ using
           observations $\tilde{u}_{t+1}$, $\tilde{u}_{t+2}$ and Eqs. (5)–(7)
6:     $t \leftarrow t + 2$
7: while $L_t(\boldsymbol{\pi}_{t+1}) \leq U_t(\boldsymbol{\pi}_{t+2}) - \epsilon$
8: return $\overline{\boldsymbol{\pi}} = (\bar{x}_t, \gamma_t(\bar{x}_t))$
```

This choice is made so as to advance the algorithm towards its termination. In particular, the M-GP-LUCB algorithm stops when, according to the confidence intervals, the strategy $\bar{x}_t$ is probably approximately better than all the others, *i.e.*, when it holds $L_t(\boldsymbol{\pi}_{t+1}) > U_t(\boldsymbol{\pi}_{t+2}) - \epsilon$. Intuitively, $\boldsymbol{\pi}_{t+1}$ represents the best candidate for being a maximin strategy profile, while $\boldsymbol{\pi}_{t+2}$ is the second-best candidate. Thus, the algorithm stops if $\boldsymbol{\pi}_{t+1}$ is better than $\boldsymbol{\pi}_{t+2}$ with sufficiently high confidence, *i.e.*, whenever the lower bound for the former is larger than the upper bound for the latter (up to an approximation of ϵ). The final strategy profile recommended by the algorithm is $\overline{\boldsymbol{\pi}} := (\bar{x}_t, \gamma_t(\bar{x}_t))$.

The following theorem shows that M-GP-LUCB is δ-PAC and provides an upper bound on the number of rounds T_δ it requires. The analysis is performed for $\epsilon = 0$, *i.e.*, when $\overline{\boldsymbol{\pi}}$ is evaluated with respect to an exact maximin profile. [3] Note that the upper bound for T_δ depends on the utility-dependent term $H^*(u) := \sum_{\boldsymbol{\pi} \in \Pi} c(\boldsymbol{\pi})$, where, for $\boldsymbol{\pi} = (x, y) \in \Pi$, $c(\boldsymbol{\pi})$ is defined as follows:

$$c(\boldsymbol{\pi}) := \frac{1}{\max\left\{(\Delta^*)^2, \left(\frac{u(x^*, y^*(x^*)) + u(x^{**}, y^*(x^{**}))}{2} - u(x, y^*(x))\right)^2\right\}},$$

where, for the ease of writing, we let $\Delta^* := u(\boldsymbol{\pi}) - u(x, y^*(x))$ and $x^{**} \in \operatorname{argmax}_{x \in \mathcal{X} \setminus \{x^*\}} u(x, y^*(x))$, *i.e.*, x^{**} is a first player's maximin strategy when x^* is removed from the available ones. This term has the same role as $H_1 := \sum_{i \in \{1, \ldots, |\Pi|\}} \frac{1}{\Delta_{(i)}^2}$ used by Audibert, Bubeck, and Munos (2010) in the best arm identification setting, where $\boldsymbol{\pi}^i$ is the i-th strategy profile in Π, which is ordered in such a way that, letting $\Delta_{(i)} := |u(\boldsymbol{\pi}^*) - u(\boldsymbol{\pi}^i)|$, it holds $\Delta_{(1)} \leq \Delta_{(2)} \leq \ldots \leq \Delta_{(|\Pi|)}$. Intuitively, $H^*(u)$ and H_1 characterize the hardness of the problem instances by determining the amount of rounds required to identify the maximin profile and the best arm, respectively.

Theorem 1. *Using a generic nondecreasing exploration term $b_t > 0$, the M-GP-LUCB algorithm stops its execution after at most T_δ rounds, where:*

$$T_\delta \leq \inf\left\{t \in \mathbb{N} \; : \; 8\, H^*(u)\, b_t\, \lambda - \frac{\lambda\, n\, m}{\sigma^2} < t\right\}. \quad (10)$$

Specifically, letting $b_t := 2 \log\left(\frac{n\, m\, \pi^2\, t^2}{6\delta}\right)$, the algorithm returns a maximin profile with confidence at least $1-\delta$, and:

$$T_\delta \leq 64\, H^*(u)\, \lambda \left(\log\left(64\, H^*(u)\, \lambda\, \pi \sqrt{\frac{n\, m}{6\delta}}\right) + \right.$$
$$\left. +2 \log\left(\log\left(64\, H^*(u) \lambda \pi \sqrt{\frac{n\, m}{6\delta}}\right)\right)\right), \quad (11)$$

where we require that $64\, \lambda\, \pi\, \sqrt{\frac{n\, m}{6\delta}} > 4.85$.

Intuitively, from the result in Theorem 1, we can infer that the most influential terms on the number of rounds required to get a specific confidence level δ are $H^*(u)$ and the noise variance λ, which impact as multiplicative constants on T_δ. On the other hand, T_δ scales only logarithmically with the number of strategy profiles $|\Pi| = n\, m$, thus allowing the execution of the M-GP-LUCB algorithm also in settings where the players have a large number of strategies available.

Algorithm 3 GP-SE(T)

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1: Initialize $\Pi_1 \leftarrow \Pi$, $\mu_0(\boldsymbol{\pi}) \leftarrow \mathbf{0}$
2: for $p = 1, 2, \ldots, P - 1$ do
3:     For each $\boldsymbol{\pi} \in \Pi_p$, query $\textsc{Sim}(\boldsymbol{\pi})$ for
           $T_p - T_{p-1}$ rounds
4:     Compute $\mu_p(\boldsymbol{\pi})$ using observations
5:     Select $\boldsymbol{\pi}_p$ according to Eqs. (12)–(13)
6:     $\Pi_{p+1} \leftarrow \Pi_p \setminus \{\boldsymbol{\pi}_p\}$
7: return the unique element $\overline{\boldsymbol{\pi}}$ of $\Pi_P$
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Fixed-Budget Setting

In the fixed-budget setting, the goal is to design δ-PAC algorithms that, given the maximum number of available rounds T (*i.e.*, the budget), find an ϵ-maximin strategy with confidence $1 - \delta_T$ as large as possible. We propose a successive elimination algorithm (called GP-SE, see Algorithm 3), which is based on an analogous method proposed by Audibert, Bubeck, and Munos (2010) for the best arm identification problem. The fundamental idea behind our GP-SE algorithm is a novel elimination rule, which is suitably defined for the problem of identifying maximin strategies.

The algorithm works by splitting the number of available rounds T into $P - 1$ phases, where, for the ease of notation, we let $P := |\Pi| = n\, m$ be the number of players' strategy profiles. At the end of each phase, the algorithm excludes from the set of candidate solutions the strategy profile that has the lowest chance of being maximin. Specifically, letting Π_p be the set of the remaining strategy profiles during phase p, at the end of p, the algorithm dismisses the strategy profile $\boldsymbol{\pi}_p := (x_p, y_p) \in \Pi_p$, defined as follows:

$$(x_p, \cdot) := \operatorname*{argmin}_{\boldsymbol{\pi} \in \Pi_p} \mu_p(\boldsymbol{\pi}), \quad (12)$$

$$y_p := \operatorname*{argmax}_{y \in \mathcal{Y} : (x_p, y) \in \Pi_p} \mu_p(x_p, y), \quad (13)$$

where μ_p represents the mean of the posterior distribution computed at the end of phase p (see Equations (2)-(4)). Intuitively, the algorithm selects the first player's strategy x_p that is less likely to be a maximin one, together with the second player's strategy y_p that is the worst for her given x_p. At the end of the last phase, the (unique) remaining strategy profile $\overline{\boldsymbol{\pi}} = (\bar{x}, \bar{y})$ is recommended by the algorithm.

[3] Assuming $\epsilon = 0$ also requires the additional w.l.o.g. assumption that the utility value of an exact maximin strategy and that one of a second-best maximin strategy are different.

Following (Audibert, Bubeck, and Munos 2010), the length of the phases have been carefully chosen so as to obtain an optimal (up to a logarithmic factor) convergence rate. Specifically, letting $\overline{\log}(P) := \frac{1}{2} + \sum_{i=2}^{P} \frac{1}{i}$, let us define $T_0 := 0$ and, for every phase $p \in \{1, \dots, P-1\}$, let:

$$T_p := \left\lceil \frac{T - P}{\overline{\log}(P)(P + 1 - p)} \right\rceil. \quad (14)$$

Then, during each phase p, the algorithm selects every remaining strategy profile in Π_p for exactly $T_p - T_{p-1}$ rounds. Let us remark that the algorithm is guaranteed to do not exceed the number of available rounds T. Indeed, each $\boldsymbol{\pi}_p$ is selected for T_p rounds, while $\overline{\boldsymbol{\pi}}$ is chosen T_{P-1} times, and $\sum_{p=1}^{P-1} T_p + T_{P-1} \leq T$ holds by definition.

The following theorem provides an upper bound on the probability δ_T that the strategy profile $\overline{\boldsymbol{\pi}}$ recommended by the GP-SE algorithm is *not* ϵ-maximin, as a function of the number of rounds T. As for the fixed-confidence setting, our result holds for $\epsilon = 0$.

Theorem 2. *Letting T be the number of available rounds, the GP-SE algorithm returns a maximin strategy profile $\boldsymbol{\pi}^*$ with confidence at least $1 - \delta_T$, where:*

$$\delta_T := 2P(n + m - 2)e^{-\frac{T-P}{8\lambda \overline{\log}(P) H_2}}, \quad (15)$$

and $H_2 := \max_{i \in \{1,\dots,P\}} i\, \Delta_{(i)}^{-2}$.

As also argued by Audibert, Bubeck, and Munos (2010), a successive elimination method provides two main advantages over a simple round robin querying strategy in which every strategy profile is queried for the same number of rounds. First, it provides a similar bound on δ_T with a better dependency on the parameters, and, second, it queries the maximin strategy profile a larger number of times, thus returning a better estimate of its expected utility.

SBGs with Infinite Strategy Spaces

We are now ready to provide our main results on SBGs with infinite strategy spaces. In the first part of the section, we show how the δ-PAC algorithms proposed in the previous sections for finite SBGs can be adapted to work with infinite strategy spaces while retaining some theoretical guarantees on the returned ϵ-maximin profiles. This requires to work with a (finite) discretized version of the original (infinite) SBGs, where the players' strategy spaces are approximated with grids made of equally spaced points. Then, in the second part of the section, we provide some results for the situations in which one cannot work with this kind of discretization, and, instead, only a limited number of points is sampled from the players' strategy spaces. This might be the case when, *e.g.*, the dimensionality d of the players' strategy spaces is too high, or there are some constraints on the strategy profiles that can be queried. Clearly, in this setting, we cannot prove δ-PAC results, as the quality of the ϵ-maximin strategy profiles inevitable depends on how the points are selected.

Let us remark that our main results rely on our assumption that the utility function u is drawn from a GP, provided some mild technical requirements are satisfied (see Assumption 1).

δ-PAC Results for Evenly-Spaced Grids

The idea is to work with a discretization of the players' strategy spaces, each made of at least K_ϵ equally spaced points, where $\epsilon \geq 0$ is the desired approximation level. This induces a new (restricted) SBGs with finite strategy spaces, where techniques presented in the previous sections can be applied. In the following, for the ease of presentation, given an SBG with infinite strategy spaces Γ, we denote with $\Gamma(K)$ the finite SBG obtained when approximating the players' strategy spaces with K equally spaced points, *i.e.*, a game in which the players have $n = m = K$ strategies available and the utility value of each of the $n\, m$ strategy profiles is the same as that one of the corresponding strategy profile in Γ. First, let us introduce the main technical requirement that we need for our results to hold.

Assumption 1 (Kernel Smoothness)**.** *A kernel $k(\boldsymbol{\pi}, \boldsymbol{\pi}')$ is said to be* smooth *over Π if, for each $L > 0$ and for some constants $a, b > 0$, the functions u drawn from* $\text{GP}(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ *satisfy:*

$$\mathbb{P}\left(\sup_{\boldsymbol{\pi} \in \Pi} \left|\frac{\partial u}{\partial x}\right| > L\right), \ \mathbb{P}\left(\sup_{\boldsymbol{\pi} \in \Pi} \left|\frac{\partial u}{\partial y}\right| > L\right) \leq a e^{-\frac{L^2}{b^2}}. \quad (16)$$

This assumption is standard when using GPs in online optimization settings (Srinivas et al. 2010), and it is satisfied by many common kernel functions for specific values of a and b, such as the squared exponential kernel and the Matérn one with smoothness parameter $\nu > 2$ (see Section for details on the definition of these kernels).

Theorem 3. *Assume that u is drawn from a* $\text{GP}(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ *satisfying Assumption 1. Given $\epsilon > 0$ and $\delta \in (0, 2)$, let $\overline{\boldsymbol{\pi}} := (\bar{x}, \bar{y}) \in \Pi$ be a maximin strategy profile for a finite game $\Gamma(K)$ where K is at least $K_\epsilon := \left\lceil \frac{b}{2\epsilon}\sqrt{\log\left(\frac{4a}{\delta}\right)} \right\rceil + 1$. Then, the following holds:*

$$\mathbb{P}\Big(|u(\boldsymbol{\pi}^*) - u(\overline{\boldsymbol{\pi}})| \leq \epsilon\Big) \geq 1 - \frac{\delta}{2}. \quad (17)$$

The following two results rely on Theorem 3 to show that the M-GP-LUCB (Algorithm 2) and the GP-SE (Algorithm 3) algorithms can be employed to find, with high confidence, ϵ-maximin strategy profiles in SBGs with infinite strategy spaces. Let us remark that, while for SBGs with finite strategy spaces our theoretical analysis is performed for $\epsilon = 0$, in the case of infinite strategy spaces it is necessary to assume a nonzero approximation level ϵ.

Corollary 1. *Assume that u is drawn from a* $\text{GP}(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ *satisfying Assumption 1. Given $\epsilon > 0$ and $\delta \in (0, 1)$, letting $b_t := 2\log\left(\frac{nm\pi^2 t^2}{3\delta}\right)$, the M-GP-LUCB algorithm applied to $\Gamma(K)$ with K at least $K_\epsilon := \left\lceil \frac{b}{2\epsilon}\sqrt{\log\left(\frac{4a}{\delta}\right)} \right\rceil + 1$ returns a strategy profile $\overline{\boldsymbol{\pi}} := (\bar{x}, \bar{y})$ such that $\mathbb{P}\left(|u(\boldsymbol{\pi}^*) - u(\bar{x}, y^*(\bar{x}))| \leq \epsilon\right) \geq 1 - \delta$. Moreover, the algorithm stops its execution after at most:*

$$T_{\delta,\epsilon} \leq 64\, H^*(u)\, \lambda \left[\log\left(64\, H^*(u)\, \lambda\, \pi\, K_\epsilon \sqrt{\frac{1}{3\delta}}\right) + \right.$$
$$\left. +2\, \log\left(\log\left(64\, H^*(u) \lambda\, \pi\, K_\epsilon \sqrt{\frac{1}{3\delta}}\right)\right)\right], \quad (18)$$

where we require that $64\, \lambda\, \pi\, K_\epsilon \sqrt{\frac{1}{3\delta}} > 4.85$.

Corollary 2. *Assume that u is drawn from a GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ satisfying Assumption 1. Given $\epsilon > 0$ and $\delta \in (0,1)$, letting T be the number of available rounds, the GP-SE algorithm applied to $\Gamma(K)$ with K at least $K_\epsilon := \left\lceil \frac{b}{2\epsilon}\sqrt{\log\left(\frac{4a}{\delta}\right)} \right\rceil + 1$ returns a profile $\overline{\boldsymbol{\pi}} := (\bar{x}, \bar{y})$ such that $\mathbb{P}\left(|u(\boldsymbol{\pi}^*) - u(\bar{x}, y^*(\bar{x}))| > \epsilon\right) < \delta_{T,\epsilon}$, where:*

$$\delta_{T,\epsilon} := 4K_\epsilon^2(K_\epsilon - 1)e^{-\frac{T-K_\epsilon^2}{8\lambda\overline{\log}(K_\epsilon^2)H_2}} + 2ae^{-\frac{b^2}{4\epsilon^2(K_\epsilon-1)^2}}. \quad (19)$$

In the result of Corollary 2, the discretization parameter K_ϵ depends on a confidence level δ that has to be chosen in advance. Another possibility is to try to minimize the overall confidence $\delta_{T,\epsilon}$ by appropriately tuning the parameter δ. Formally, a valid confidence level can be defined as follows:

$$\delta_{\text{opt}} := \inf\{\delta \in (0,1) \ : \ \delta_{T,\epsilon}\}, \quad (20)$$

noticing that $\delta_{T,\epsilon}$ depends on δ also through the term K_ϵ. Unfortunately, this minimization problem does not admit a closed-form optimal solution. Nevertheless, we can compute an (approximate) optimal value for δ by employing numerical methods (Nocedal and Wright 2006).

Arbitrary Discretization

Whenever using an equally-spaced grid as a discretization scheme is unfeasible, the theoretical results based on Theorem 3 do not hold anymore. Nevertheless, given any finite sets of players' strategies, we can bound with high probability the distance of a maximin profile $\boldsymbol{\pi}^*$ from the strategy profile learned in the resulting (finite) discretized SBG. Formally, let $\mathcal{X}_n \subseteq \mathcal{X}$ be a finite set of n first player's strategies and, similarly, let $\mathcal{Y}_m \subseteq \mathcal{Y}$ be a finite set of m second player's strategies. Thus, the resulting finite SBG $\Gamma := (\mathcal{X}_n, \mathcal{Y}_m, u)$ has $n\,m$ strategy profiles. Let $d_x^{\max} := \max_{x \in \mathcal{X}} \min_{x_i \in \mathcal{X}_n} |x - x_i|$, $d_y^{\max} := \max_{y \in \mathcal{Y}} \min_{y_i \in \mathcal{Y}_m} |y - y_i|$, then, we can show the following result.

Theorem 4. *Assume that u is drawn from a* GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$ *satisfying Assumption 1. Given $\delta \in (0,2)$, let $\overline{\boldsymbol{\pi}} := (\bar{x}, \bar{y}) \in \mathcal{X}_n \times \mathcal{Y}_m$ be a maximin strategy profile for a finite game $\Gamma := (\mathcal{X}_n, \mathcal{Y}_m, u)$. Then, the following holds:*

$$\mathbb{P}\left(|u(\boldsymbol{\pi}^*) - u(\overline{\boldsymbol{\pi}})| \leq b\sqrt{\log\frac{4a}{\delta}}\max\left\{d_x^{\max}, d_y^{\max}\right\}\right) \geq 1 - \frac{\delta}{2}.$$

Let us remark that the result in Theorem 4 can be applied any time using an equally-spaced grid as a discretization scheme is unfeasible, as it is the case, *e.g.*, when the dimensionality d of the players' strategy spaces is too large.

Experimental Results

We experimentally evaluate our algorithms on both finite and infinite SBGs. As for the finite case, we compare the performances (with different metrics) of our M-GP-LUCB and GP-SE algorithms against two baselines. The first one is the M-LUCB algorithm proposed by Garivier, Kaufmann, and Koolen (2016), which is the state of the art for learning maximin strategies in finite SBGs and can be easily adapted to our setting by using a different exploration term b_t. We introduce a second baseline to empirically evaluate how our algorithms speed up their convergence by leveraging correlation of the utilities. Specifically, it is a variation of our M-GP-LUCB algorithm (called M-G-LUCB) where utility values are assumed drawn from independent Gaussian random variables, instead of a GP. [4] As for SBGs with infinite strategy spaces, there are no state-of-the-art techniques that we can use as a baseline for comparison. Thus, we show the quality (in terms of ϵ) of the strategy profiles returned by our algorithms using different values of K_ϵ for the discretized games. The average ϵ values obtained empirically (called $\hat{\epsilon}$ thereafter) are compared against the theoretical values prescribed by Theorem 3 (for the given K_ϵ), so as to evaluate whether our bounds are strict or not.

Random Game Instances

As for finite SBGs, we test the algorithms on random instances generated by sampling from GP$(\mathbf{0}, k(\boldsymbol{\pi}, \boldsymbol{\pi}'))$, using the following two commonly used kernel functions (see (Williams and Rasmussen 2006) for more details):

- *squared exponential*: $k(\boldsymbol{\pi}, \boldsymbol{\pi}') := e^{-\frac{1}{2l^2}||\boldsymbol{\pi}-\boldsymbol{\pi}'||^2}$, where l is a length-scale parameter;
- *Matérn*: $k(\boldsymbol{\pi}, \boldsymbol{\pi}') := \frac{2^{1-\nu}}{G(\nu)} r^\nu B_\nu(r)$, where $r := \frac{\sqrt{2\nu}}{l}||\boldsymbol{\pi} - \boldsymbol{\pi}'||$, ν controls the smoothness of the functions, l is a length-scale parameter, B_ν is the second-kind Bessel function, and G is the Gamma function.

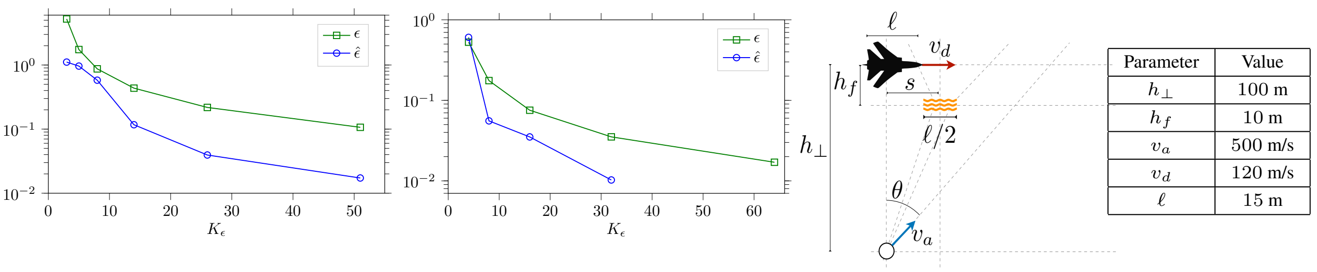
We set $l \in \{0.1, 2\}$ and $\nu \in \{1.5, 2.5\}$, generating 30 instances for each possible combination of kernel function and parameter values. As for SBGs with infinite strategy spaces, we test on instances generated from distributions with $l = 0.1$ and, with the Matérn kernel, $\nu \in \{1.5, 2.5\}$. The infinite strategy spaces are approximated with a discretization on a grid made of 100 equally-spaced points.

In the fixed-confidence setting, we let $\delta = 0.1$ and stop the algorithms after $T \in \{30\text{k}, 100\text{k}\}$ rounds. Similarly, the GP-SE algorithm is run with a budget $T \in \{30\text{k}, 100\text{k}\}$. For each possible combination of algorithm, game instance, and round-limit T, we average the results over 100 runs.

The results on finite SBGs are reported in Table 1, where T_δ is the average number of queries used by the algorithm in the runs not exceeding the round-limit T, $\%end$ is the percentage of runs the algorithm terminates before T rounds, and $\%opt$ is the percentage of runs the algorithm is able to correctly identify the maximin profile $\boldsymbol{\pi}^*$. Notice that M-GP-LUCB and M-G-LUCB clearly outperform M-LUCB, as the latter requires a number of rounds T_δ an order of magnitude larger. M-GP-LUCB and M-G-LUCB provide similar performances in terms of T_δ, but the former identifies the maximin profile more frequently than the latter. While always using the maximum number of rounds T, GP-SE is the best algorithm in identifying the maximin profile.

As for infinite SBGs, Figure 1(*Left*) provides the values of ϵ and $\hat{\epsilon}$ for an instance generated from a Matérn kernel with $\nu = 2.5$ (see Appendix E in the full version (Marchesi, Trovò, and Gatti 2019) for more results). In all the instances, $\hat{\epsilon}$ is lower than ϵ, empirically proving the correctness of our guarantees. Moreover, as expected, $\hat{\epsilon}$ decreases as the number of discretization points K_ϵ increases.

[4] The formulas for updating the mean μ_t and the variance σ_t^2 of the posterior distribution are changed accordingly.



Parameter	Value
$h_\perp$	100 m
h_f	10 m
v_a	500 m/s
v_d	120 m/s
ℓ	15 m

Figure 1: (*Left*) ϵ vs. $\hat{\epsilon}$ for different values of K_ϵ (Matérn kernel with $\nu = 2.5$). (*Center*) ϵ vs. $\hat{\epsilon}$ for different values of K_ϵ (*Hit-the-Spitfire* security game). (*Right*) *Hit-the-Spitfire* security game instance and values for its parameters used in the experiments.

Table 1: Experimental results of algorithms M-LUCB, M-G-LUCB, and M-GP-LUCB on SBGs with finite strategy spaces.

			M-LUCB			M-G-LUCB			M-GP-LUCB			GP-SE
		T	T_δ	$\%end$	$\%opt$	T_δ	$\%end$	$\%opt$	T_δ	$\%end$	$\%opt$	$\%opt$
SQE	$l = 0.1$	30k	10673.86	53.33	87.13	227.23	96.70	86.73	229.19	93.33	86.66	100.00
	$l = 2.0$	30k	23788.96	13.73	84.80	2460.19	89.23	77.73	2020.89	76.66	93.36	93.23
		100k	42103.86	46.66	88.63	3535.00	91.86	78.56	5656.77	76.77	93.30	96.60
$M_{1.5}$	$l = 0.1$	30k	13869.59	56.06	66.66	222.06	100.00	66.83	224.87	100.00	66.66	100.00
	$l = 2.0$	30k	18978.75	33.33	76.26	2532.98	91.30	76.90	3775.65	88.03	78.86	98.30
		100k	28798.42	50.00	80.70	3662.00	95.36	77.00	4618.27	93.33	79.53	98.76
$M_{2.5}$	$l = 0.1$	30k	13335.26	79.80	86.66	168.61	97.90	86.06	171.62	96.66	86.66	99.83
	$l = 2.0$	30k	20404.41	24.66	89.26	1984.55	92.10	86.63	2435.84	88.24	95.27	95.60
		100k	49198.82	62.06	92.86	2617.31	93.70	87.13	2626.89	86.66	94.93	96.46

Security Game Instances

We also test on a SBG instance with infinite strategy spaces inspired by a real-world security game., called *Hit-the-Spitfire*. This game models a military scenario in which a terrestrial counter-air defensive unit has to fire a heat-seeking missile to an approaching enemy airplane, which, after the missile has been launched, can deploy an obfuscating flare so as to try to deflect it. The model underlying such game and the parameters used in the experiment are depicted in Figure 1(*Right*), where $h_\perp$ is the distance between the airplane and the terrestrial unit, h_f is the distance of the flare from the plane, v_a and v_d are the speed of the missile and the plane, respectively, while ℓ is the length of the plane, with the flare covering half of this space ($\frac{\ell}{2}$). The first player (the counter-air defensive unit) can determine the angle $\theta \in [0, 1]$ (radians) at which the missile is launched, while the second player (the airplane) has to decide the position $s \in [0, s_{\max}]$ where to release the flare. If the missile hits the plane, then it incurs damage $d \in \mathbb{R}^+$ that depends on the hitting point (the nearer to the center of the plane, the higher). If the missile hits the flare, then there is some probability that it is deflected away from the airplane, otherwise, the missile still hits the target. The probability of deflection is large when the distance of the airplane from the deployed flare is larger. [5] We run the M-GP-LUCB with $\delta = 0.1$.

Figure 1(*Center*) reports the results of running the M-GP-LUCB algorithm with $\delta = 0.1$ on the *Hit-the-Spitfire* game (performing 100 runs for each K_ϵ). Notice that, in most of the cases, $\hat{\epsilon}$ is lower than the theoretical value ϵ. This is unexpected, since, in this setting, the assumption that the utility function u is drawn from a GP does not hold. We remark that, in all the runs, M-GP-LUCB is able to identify the maximin strategy profile over the given grid.

Discussion and Future Works

To the best of our knowledge, we provided the first learning algorithms for infinite SBGs enjoying δ-PAC theoretical guarantees on the quality of the returned solutions. This significantly advances the current state of the art for SBGs, as dealing with infinite strategies paves the way to the application of such models in complex real-world settings. The fundamental ingredient of our results is the assumption that the utility functions are drawn from a GP, which allows us to encode function regularities without relying on specific parametric assumptions, such as, *e.g.*, linearity.

In future, we will address the case of general (*i.e.*, non-zero-sum and multi-player) SBGs with finite (or even infinite) strategy spaces, where one seeks for an (approximate) Nash equilibrium. An interesting question is how to generalize our learning algorithms based on best arm identification techniques to deal with Nash-equilibrium conditions instead of maximin ones. This would pave the way to the application of our techniques to other interesting problems, such as multi-agent evaluation by means of meta-games (Tuyls et al. 2018; Rowland et al. 2019). Finally, the use of more sophisticated bounds might, *e.g.*, the ones presented by (Nuara et al. 2020), might improve the M-GP-LUCB algorithm.

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[5] See (Marchesi, Trovò, and Gatti 2019) for more details.

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